MODEL PAPER - I

MATHEMATICS

Time allowed : 3 hours | Maximum marks : 100

General Instructions

1. All questions are compulsory.
2. The question paper consists of 29 questions divided into three sections A, B and C. Section A comprises of 10 questions of one mark each, Section B comprises of 12 questions of four marks each and Section C comprises of 7 questions of six marks each.
3. All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
4. There is no overall choice. However, internal choice has been provided in 4 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.
5. Use of calculators is not permitted.

SECTION A

Question number 1 to 10 carry one mark each.

1. Find the value of x, if

$$\begin{pmatrix} 5x + y & -y \\ 2y - x & 3 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -3 & 3 \end{pmatrix}$$

2. Let * be a binary operation on N given by a * b = HCF (a, b), a, b $\in$ N. Write the value of 6 * 4.

3. Evaluate : $\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1 - x^2}} dx$

4. Evaluate : $\int \frac{\sec^2(\log x)}{x} dx$

5. Write the principal value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$.

6. Write the value of the determinant :

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

7. Find the value of x from the following :

$$\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0$$

8. Find the value of $(\hat{i} \times \hat{j}) \cdot \hat{k} + (\hat{j} \times \hat{k}) \cdot \hat{i} + (\hat{k} \times \hat{i}) \cdot \hat{j}$

9. Write the direction cosines of the line equally inclined to the three coordinate axes.

10. If $\vec{p}$ is a unit vector and $(\vec{x} - \vec{p}).(\vec{x} + \vec{p}) = 80$, then find $|\vec{x}|$.

SECTION B

Question numbers 11 to 22 carry 4 marks each.

11. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When x = 8 cm and y = 6 cm, find the rate of change of (a) the perimeter, (b) the area of the rectangle.

OR

Find the intervals in which the function f given by f(x) = sin x + cos x, $0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

OR

12. If $(\cos x)^y = (\sin y)^x$, find $\frac{dy}{dx}$.

13. Consider $f : R - \left\{ \frac{-4}{3} \right\} \rightarrow R - \left\{ \frac{4}{3} \right\}$ defined as $f(x) = \frac{4x}{3x + 4}$

Show that f is invertible. Hence find f^{-1}.

14. Evaluate : $\int \frac{dx}{\sqrt{5 - 4x - 2x^2}}$.

OR

Evaluate : $\int x \sin^{-1} x \, dx$.

15. If $y = \frac{\sin^{-1} x}{\sqrt{1 - x^2}}$, show that $(1 - x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$.

16. In a multiple choice examination with three possible answers (out of which only one is correct) for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

17. Using properties of determinants, prove the following :

$$\begin{vmatrix} a + b + c & -c & -b \\ -c & a + b + c & -a \\ -b & -a & a + b + c \end{vmatrix} = 2(a + b)(b + c)(c + a)$$

18. Solve the following differential equation :

$$x\frac{dy}{dx} = y - x \tan\left(\frac{y}{x}\right).$$

19. Solve the following differential equation :

$$\cos^2 x \cdot \frac{dy}{dx} + y = \tan x.$$

20. Find the shortest distance between the lines

□ □□
r □ □1 □ □□ □ □2 □ □□ □ □□ □ 1□ k
□ i □ j

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k}).$$

21. Prove the following :

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]=\frac{x}{2},\ x\in\left(0,\frac{\pi}{4}\right).$$

OR

Solve for x :

$$2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$$

22. The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ.

OR

$\vec{a}$, $\vec{b}$ and $\vec{c}$ are three coplanar vectors. Show that $(\vec{a}+\vec{b})$, $(\vec{b}+\vec{c})$ and $(\vec{c}+\vec{a})$ are also coplanar.

SECTION C

Question number 23 to 29 carry 6 marks each.

23. Find the equation of the plane determined by the points A (3, –1, 2), B (5, 2, 4) and C (–1, –1, 6). Also find the distance of the point P(6, 5, 9) from the plane.

24. Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

25. Evaluate : $\displaystyle\int_0^{\pi}\frac{x\,dx}{a^2\cos^2 x+b^2\sin^2 x}$.

26. Using matrices, solve the following system of equation :

$$x + y + z = 6$$

$$x + 2z = 7$$

$$3x + y + z = 12$$

OR

Obtain the inverse of the following matrix using elementary operations:

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ & 4 & 1 \end{bmatrix}.$$

27. Coloured balls are distributed in three bags as shown in the following table :

Bag	Colour of the Ball		
	Red	White	Black
I	1	2	3
II	2	4	1
III	4	5	3

A bag is selected at random and then two balls are randomly drawn from the selected bag. They happen to be black and red. What is the probability that they came from bag I?

28. A dealer wishes to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space for at most 20 items. A fan costs him Rs. 360 and a sewing machine Rs. 240. His expectation is that he can sell a fan at a profit of Rs. 22 and a sewing machine at a profit of Rs. 18. Assuming that be can sell all the items that he can buy, how should he invest his money in order to maximise the profit? Formulate this as a linear programming problem and solve it graphically.

29. If the sum of the lengths of the hypotenuse and a side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

OR

A tank with rectangular base and rectangular sides open at the top is to be constructed so that it's depts is 2m and volume is $8m^3$. If building of tank cost Rs. 70 per sq meter for the base and Rs. 45 per sq meter for the sides. What is the cost of least expensive tank.

SOLUTIONS AND MARKING SCHEME

SECTION A

Note : For 1 mark questions in Section A, full marks are given if answer is correct (i.e. the last step of the solution). Here, solution is given for your help.

Marks

1. We are given

$$\begin{bmatrix} 5x + y & -y \\ 2y - x & 3 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ -5 & 3 \end{bmatrix}$$

$\therefore$ $5x + y = 4$ and $-y = 1$

$\therefore$ $y = -1$ and $5x - 1 = 4$

or $5x = 5$

$\therefore$ $x = 1$ ...(1)

2. $6 * 4 =$ HCF of 6 and $4 = 2$. ...(1)

3. $$\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1 - x^2}} dx = \left| \sin^{-1} x \right|_0^{1/\sqrt{2}}$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1} 0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$ ...(1)

4. Let $I = \int \frac{\sec^2 (\log x)}{x} dx$

Let $\log x = t$

then $\frac{1}{x} dx = dt$

or $\quad dx = x\ dt$

$\therefore \quad I = \int \sec^2 t\ dt$

$= \tan t + c$

$= \tan(\log x) + c$ ...(1)

5. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{5\pi}{6}\right)\right]$

$= \cos^{-1}\left[\cos\left(\frac{5\pi}{6}\right)\right]$

$= \frac{5\pi}{6}$ ...(1)

6. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix}$

$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$

$= 0$ ...(1)

7. Here $\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0$

or $\quad 2x^2 - 8 = 0$

or $\quad x^2 - 4 = 0$

$x = \pm 2$ ...(1)

8. $(\hat{i} \times \hat{j}) \cdot \hat{k} + (\hat{j} \times \hat{k}) \cdot \hat{i} + (\hat{k} \times \hat{i}) \cdot \hat{j}$

$= \hat{k} \cdot \hat{k} + \hat{i} \cdot \hat{i} + \hat{j} \cdot \hat{j}$

$= 1 + 1 + 1$

$= 3$ ...(1)

9. The d.c. of a line equally inclined to the coordinate axes are $\left(\frac{\pm 1}{\sqrt{3}}, \frac{\pm 1}{\sqrt{3}}, \frac{\pm 1}{\sqrt{3}}\right)$. ...(1)

10. $$(\vec{x} - \vec{p}).(\vec{x} + \vec{p}) = 80$$

$\therefore$ $$|\vec{x}|^2 - |\vec{p}|^2 = 80$$

As $\vec{p}$ is a unit vector,

$$|\vec{p}| = 1$$

$\therefore$ $$|\vec{x}|^2 - 1 = 80$$

or $$|\vec{x}|^2 = 81$$

$\therefore$ $$|x| = 9$$ ...(1)

SECTION B

11. Let P be the perimeter and A be the area of the rectangle at any time t, then

$$P = 2(x + y) \text{ and } A = xy$$

It is given that $\frac{dx}{dt} = -5$ cm/minute

and $\frac{dy}{dt} = 4$ cm/minute ...(1)

(i) We have $P = 2(x + y)$

$\therefore$ $$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

Marks

$= 2\ (-5 + 4)$

$= -2$ cm/minute ...(1½)

(ii) We have $A = xy$

$\therefore \quad \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$

$= [8 \times 4 + 6(-5)]$ ......($\because$ x = 8 and y = 6)

$= (32 - 30)$

$= 2$ cm^2/minute ...(1½)

OR

The given function is

$$f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$$

$$\therefore \quad f'(x) = \cos x - \sin x$$

$$= -\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x\right)$$

$$= -\sqrt{2}\left(\sin x \cos\frac{\pi}{4} - \cos x \sin\frac{\pi}{4}\right)$$

$$= -\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) \quad ...(1)$$

For strictly decreasing function,

$$f'(x) < 0$$

$$\therefore \quad -\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) < 0$$

or $$\sin\left(x - \frac{\pi}{4}\right) > 0$$

or $$0 < x - \frac{\pi}{4} < \pi$$

Marks

or $$\frac{\pi}{4} < x < \pi + \frac{\pi}{4}$$

or $$\frac{\pi}{4} < x < \frac{5\pi}{4}$$

Thus f(x) is a strictly decreasing function on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ ...(2)

As sin x and cos x are well defined in $[0, 2\pi]$,

f (x) = sin x + cos x is an increasing function in the complement of interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

i.e., in $$\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$$ ...(1)

12. We are given

$$(\cos x)^y = (\sin y)^x$$

Taking log of both sides, we get

$$y \log \cos x = x \log \sin y$$ ...(½)

Differentiating w.r.t. x, we get

$$y \cdot \frac{1}{\cos x} \cdot (-\sin x) + \log \cos x \cdot \frac{dy}{dx} = x \cdot \frac{1}{\sin y} \cdot (\cos y)\frac{dy}{dx} + \log \sin y \cdot 1$$ ...(2)

or $$-y \tan x + \log \cos x \frac{dy}{dx} = x \cot y \frac{dy}{dx} + \log \sin y$$

$$\Rightarrow \frac{dy}{dx}(\log \cos x - x \cot y) = \log \sin y + y \tan x$$ ...(1)

$$\therefore \frac{dy}{dx} = \frac{\log \sin y + y \tan x}{\log \cos x - x \cot y}$$ ...(½)

Marks

13. For showing f is one-one ...(1½)

For showing f is onto ...(1)

As f is one-one and onto f is invertible ...(½)

For finding $f^{-1}(x) = \frac{4x}{4 - 3x}$ ...(1)

14. Let $I = \int \frac{dx}{\sqrt{5 - 4x - 2x^2}}$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{5}{2} - 2x - x^2}}$ ...(½)

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{5}{2} - \left(2x + x^2 + 1 - 1\right)}}$

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{7}{2} - (x + 1)^2}}$ ...(1½)

$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\frac{7}{2}\right)^2 - (x + 1)^2}}$

$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x + 1}{\frac{\sqrt{7}}{\sqrt{2}}}\right) + c$

$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{\sqrt{2}\,(x + 1)}{\sqrt{7}}\right) + c$ ...(2)

OR

Let $\quad I = \int x \sin^{-1} x \, dx$

$$= \int \sin^{-1} x . x \, dx$$

$$= \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{1}{\sqrt{1 - x^2}} \cdot \frac{x^2}{2} dx \qquad ...(1)$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{-x^2}{\sqrt{1 - x^2}} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{1 - x^2 - 1}{\sqrt{1 - x^2}} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \sqrt{1 - x^2} dx - \frac{1}{2} \int \frac{dx}{\sqrt{1 - x^2}} \qquad ...(1)$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \cdot \left[\frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} x \right] - \frac{1}{2} \sin^{-1} x + c$$

...(1)

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{x}{4} \sqrt{1 - x^2} + \frac{1}{4} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + c$$

$$= \frac{x^2 \sin^{-1} x}{2} - \frac{1}{4} \sin^{-1} x + \frac{x}{4} \sqrt{1 - x^2} + c \qquad ...(1)$$

15. We have

$$y = \frac{\sin^{-1} x}{\sqrt{1 - x^2}}$$

$$\Rightarrow \qquad y\sqrt{1-x^2} = \sin^{-1} x$$

Differentiating w.r.t. x, we get

$$y.\frac{(-2x)}{2\sqrt{1-x^2}} + \sqrt{1-x^2}\,\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

or $$-xy + \left(1-x^2\right)\frac{dy}{dx} = 1$$ ...(1½)

Differentiating again,

$$-x\frac{dy}{dx} - y + \left(1-x^2\right)\frac{d^2y}{dx^2} + \frac{dy}{dx}(-2x) = 0$$

or $$\left(1-x^2\right)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$$ ...(2½)

which is the required result.

16. Here $p = \frac{1}{3}$, $q = 1 - \frac{1}{3} = \frac{2}{3}$, and $n = 5$

Let x denote the number of correct answers. ...(1)

Probability of r successes is given by

$$P(X = r) = {}^nC_r p^r \; q^{n-r}, \; r = 1, 2, 3.....$$

$\therefore \quad P(X = 4 \text{ or } 5) = P(X = 4) + P\ (X = 5)$ ...(½)

$$= {}^5C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^1 + {}^5C_5\left(\frac{1}{3}\right)^5$$ ...(1)

$$= 5\cdot\frac{2}{3^5} + 1\cdot\frac{1}{3^5}$$

$$= \frac{10}{243} + \frac{1}{243} = \frac{11}{243}.$$ ...(1½)

Marks

17. LHS $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$

$R_1 \to R_1 + R_2$

$R_2 \to R_2 + R_3$

$$\begin{vmatrix} a+b & a+b & -(a+b) \\ -(b+c) & b+c & b+c \\ -b & -a & a+b+c \end{vmatrix}$$ ...(2)

$$= (a+b)(b+c)\begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ b & -a & a+b+c \end{vmatrix}$$ ...(½)

$C_1 \to C_1 + C_3$

$$= (a+b)(b+c)\begin{vmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ c+a & -a & a+b+c \end{vmatrix}$$ ...(1)

$= 2(a+b)(b+c)(c+a)$ ...(½)

18. The given differential equation is

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$$

or $$\frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right)$$

Let $$y = zx$$

$\therefore$ $$\frac{dy}{dx} = z + x\frac{dz}{dx}$$ ...(1)

Marks

$\therefore \quad z + x\frac{dz}{dx} = z - \tan z$

or $\quad x\frac{dz}{dx} = -\tan z$ ...(1)

or $\quad \int \cot z \, dz + \int \frac{dx}{x} = 0$ ...(½)

$\therefore \quad \log \sin z + \log x = \log c$

or $\quad \log (x \sin z) = \log c$ ...(1)

or $\quad x \sin\left(\frac{y}{x}\right) = c$ ...(½)

which is the required solution.

19. The given differential equation is

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

or $\quad \frac{dy}{dx} + \sec^2 x . y = \tan x . \sec^2 x$

It is a linear differential equation

Integrating factor $= e^{\int \sec^2 x \, dx} = e^{\tan x}$ ...(1)

$\therefore$ Solution of the differential equation is

$$y.e^{\tan x} = \int e^{\tan x} . \tan x \sec^2 x \, dx + c$$ ...(½)

Now, we find $\quad I_1 = \int e^{\tan x} \cdot \tan x \sec^2 x \, dx$

Let $\quad \tan x = t, \sec^2 x \, dx = dt$

$\therefore \qquad I_1 = \int te^t \, dt$

$= t.e^t - \int e^t dt$

$= t \,.\, e^t - e^t$

$= (t - 1)e^t = (\tan x - 1)\, e^{\tan x}$ ...(2)

$\therefore$ From (i), solution is

$$y \,.\, e^{\tan x} = (\tan x - 1)\, e^{\tan x} + c$$

or $\qquad y = (\tan x - 1) + ce^{-\tan x}$ ...(½)

20. Equations of the two lines are :

$$\overrightarrow{r} = (1 + \lambda)\hat{i} + (2 - \lambda)\hat{j} + (\lambda + 1)\hat{k}$$

or $\qquad \overrightarrow{r} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) + \lambda\left(\hat{i} - \hat{j} + \hat{k}\right)$ ...(i)

and $\qquad \overrightarrow{r} = \left(2\hat{i} - \hat{j} - \hat{k}\right) + \mu\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$ ...(ii)

Here $\overrightarrow{a_1} = \hat{i} + 2\hat{j} + \hat{k}$ and $\overrightarrow{a_2} = 2\hat{i} - \hat{j} - \hat{k}$

and $\overrightarrow{b_1} = \hat{i} - \hat{j} + \hat{k}$ and $\overrightarrow{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$ ...(1)

$\therefore \qquad \overrightarrow{a_2} - \overrightarrow{a_1} = \left(2\hat{i} - \hat{j} - \hat{k}\right) - \left(\hat{i} + 2\hat{j} + \hat{k}\right)$

$= \hat{i} - 3\hat{j} - 2\hat{k}$ ...(½)

and $\qquad \overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} \hat{i} & \hat{i} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$

$= \hat{i}\,(-3) - \hat{j}\,(0) + \hat{k}\,(3)$

$= -3\hat{i} + 3\hat{k}$ ...(1)

$\therefore \quad \left|\vec{b_1} \times \vec{b_2}\right| = \sqrt{9 + 9} = 3\sqrt{2}$

$\therefore$ S.D. between the lines $= \dfrac{\left|\left(\vec{a_2} - \vec{a_1}\right) \cdot \left(\vec{b_1} \times \vec{b_2}\right)\right|}{\left|\vec{b_1} \times \vec{b_2}\right|}$ ...(½)

$= \dfrac{\left|\left(\hat{i} - 3\hat{j} - 2\hat{k}\right) \cdot \left(-3\hat{i} + 3\hat{k}\right)\right|}{3\sqrt{2}}$

$= \dfrac{|-3 - 6|}{3\sqrt{2}}$

$= \dfrac{9}{3\sqrt{2}} = \dfrac{3}{\sqrt{2}}$ units ...(1)

21. $\cot^{-1}\left[\dfrac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}\right]$

$= \cot^{-1}\left[\dfrac{\sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2} + \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}{\sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2} - \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}\right]$ ...(1)

$\ldots\left[\because x \in \left(0, \frac{\pi}{4}\right)\right]$

$= \cot^{-1}\left[\dfrac{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) + \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) - \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}\right]$ ...(1)

$= \cot^{-1}\left(\dfrac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)$ ...(1)

$= \cot^{-1}\left[\cot\left(\frac{x}{2}\right)\right]$

$= \dfrac{x}{2}$ ...(1)

Marks

OR

The given equation is

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x)$ ...(1½)

$\Rightarrow \quad \dfrac{2\cos x}{\sin^2 x} = 2\operatorname{cosec} x$ ...(1)

$\Rightarrow \quad \cos x = \operatorname{cosec} x \,.\, \sin^2 x$

$\Rightarrow \quad \cos x = \sin x$

$\therefore \quad x = \dfrac{\pi}{4}$ ...(1½)

22. Unit vector along the sum of vectors

$$\vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k} \text{ and } \vec{b} = \lambda\hat{i} + 2\hat{j} + 3\hat{k} \text{ is}$$

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} = \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 6^2 + (-2)^2}}$$

$= \dfrac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}}$ ...(1½)

We are given that dot product of above unit vector with the vector $\hat{i} + \hat{j} + \hat{k}$ is 1.

$\therefore \quad \dfrac{(2+\lambda)}{\sqrt{\lambda^2+4\lambda+44}} \cdot 1 + \dfrac{6}{\sqrt{\lambda^2+4\lambda+44}} - \dfrac{2}{\sqrt{\lambda^2+4\lambda+44}} = 1$ ...(1)

or $\quad 2 + \lambda + 6 - 2 = \sqrt{\lambda^2 + 4\lambda + 44}$

or $\quad (\lambda + 6)^2 = \lambda^2 + 4\lambda + 44$

or $\quad \lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 44$

Marks

or $\quad 8\lambda = 8$

or $\quad \lambda = 1$ ...(1½)

OR

$\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar $\Rightarrow$ $\vec{a}\ \vec{b}\ \vec{c} = 0$ ...(½)

$\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar if

$\vec{a} + \vec{b}\quad \vec{b} + \vec{c}\quad \vec{c} + \vec{a} = 0$ (½)

For showing

$$\vec{a} + \vec{b}\quad \vec{b} + \vec{c}\quad \vec{c} + \vec{a} = 2\ \vec{a}\ \vec{b}\ \vec{c} \qquad (3)$$

SECTION C

23. Equation of the plane through the points A (3, – 1, 2), B (5, 2, 4) and C (–1, –1, 6) is

$$\begin{vmatrix} x-3 & y+1 & z-2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 0 \qquad ...(2½)$$

i.e. $\quad 3x - 4y + 3z = 19$ ...(1½)

Distance of point (6, 5, 9) from plane $3x - 4y + 3z = 19$

$$= \frac{|18 - 20 + 27 - 19|}{\sqrt{9 + 16 + 9}}$$

$$= \frac{6}{\sqrt{34}} \text{ units} \qquad ...(1)$$

24. The given parabola is $y^2 = x$ ....(i)

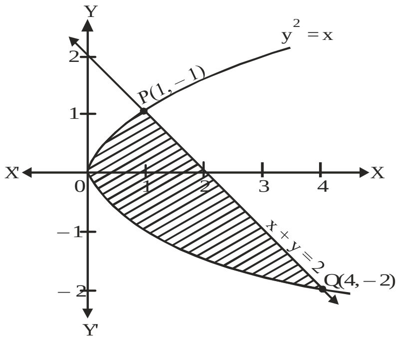
It represents a parabola with vertex at O (0, 0)

Marks

The given line is

$$x + y = 2$$

or $$x = 2 - y \qquad ....(ii)$$



...(1)

Solving (i) and (ii), we get the point of intersection P (1, 1) and Q (4, – 2) ...(1)

Required area = Area of the shaded region

$$= \int_{-2}^{1} \left[(2 - y) - y^2\right] dy \qquad ...(2)$$

$$= \left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right)_{-2}^{1} \qquad ...(1)$$

$$= \left[\left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right)\right]$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3} + 4 + 2 - \frac{8}{3}\right)$$

$$= \frac{12 - 3 - 2 + 24 + 12 - 16}{6}$$

$$= \frac{27}{6}$$

$$= \frac{9}{2} \text{ sq. units} \qquad ...(1)$$

25. Let $I = \int_0^{\pi} \frac{x\ dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

or $I = \int_0^{\pi} \frac{(\pi - x)dx}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)}$

or $I = \int_0^{\pi} \frac{(\pi - x)dx}{a^2 \cos^2 x + b^2 \sin^2 x}$ ....(ii)

Adding (i) and (ii), we get

$2I = \pi\int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$ ....(iii) ...(1)

or $2I = \pi.2\int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

Using property $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$, If $f(2a - x) = f(x)$ ...(1)

or $I = \pi\int_0^{\pi/2} \frac{\sec^2 x\ dx}{a^2 + b^2 \tan^2 x}$ ...(1)

Let tan x = t then $\sec^2 x\ dx = dt$

When x = 0, t = 0 and when $x \to \frac{\pi}{2}, t \to \infty$

$\therefore$ $I = \pi\int_0^{\infty} \frac{dt}{a^2 + b^2 t^2}$ ...(1)

$= \frac{\pi}{b^2}\int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$

Marks

$$= \frac{\pi}{b^2} \cdot \frac{1}{a/b} \left[\tan^{-1} \frac{t}{a/b} \right]_0^{\infty} \qquad ...(1)$$

$$= \frac{\pi}{ab} \left[\tan^{-1} \frac{bt}{a} \right]_0^{\infty}$$

$$= \frac{\pi}{ab} \left[\frac{\pi}{2} \right]$$

$$= \frac{\pi^2}{2ab} \qquad ...(1)$$

26. The given system of equations is

$$x + y + z = 6$$

$$x + 2z = 7$$

$$3x + y = z = 12$$

or $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

or $X = A^{-1}B$ ...(i) ...(1)

Now

$$|A| = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= 1\,(-2) - 1\,(-5) + 1\,(1)$$

$$= -2 + 5 + 1$$

$$= 4 \neq 0 \Rightarrow A^{-1} \text{ exists.} \qquad ...(1)$$

Now cofactors of elements of Matrix A are :

$A_{11} = (-1)^2\ (-2) = -2,$ $A_{12} = (-1)^3\ (-5) = 5,$ $A_{13} = (-1)^4\ (I) = 1$

$A_{21} = (-1)^3 (0) = 0,$ $\quad A_{22} = (1)^4 (-2) = -2,$ $\quad A_{23} = (-1)^5 (-2) = 2$

$A_{31} = (-1)^4 (2) = 2,$ $\quad A_{32} = (-1)^5 (1) = -1,$ $\quad A_{33} = (-1)^6 (-1) = -1$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix} \quad \ldots(2)$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} \Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix} \quad \ldots(½)$$

$\therefore$ From (i), $X = A^{-1} B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -12 + 0 + 24 \\ 30 - 14 - 12 \\ 6 + 14 - 12 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore \quad x = 3, \; y = 1 \text{ and } z = 2 \quad \ldots(1½)$

OR

26. By using elementary row transformations, we can write

$$A = IA$$

Marks

i.e., $$\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$ ...(1)

Applying $R_1 \to R_2 - R_2$, we get

$$\begin{bmatrix} 1 & -3 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$ ...(1)

Applying $R_2 \to R_2 - 2R_1$, we get

$$\begin{bmatrix} 1 & -3 & -1 \\ 0 & 9 & 2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$ ...(1)

Applying $R_1 \to R_1 + R_3$, we get

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 9 & 2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$ ...(½)

Applying $R_2 \to R_2 - 2R_3$, we get

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix} A$$ ...(½)

Applying $R_1 \to R_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix} A$$ ...(½)

Applying $R_3 \to R_3 - 4R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & 12 & 9 \end{bmatrix} A \qquad ...(½)$$

$$A^{-1} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix} \qquad ...(1)$$

27. Let the events be

E_1 : Bag I is selected

E_2 : Bag II is selected

E_3 : Bag III is selected

and A : a black and a red ball are drawn ...(1)

$\therefore \quad P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$ ...(1)

$$P(A/E_1) = \frac{1 \times 3}{{}^6C_2} = \frac{3}{15} = \frac{1}{5}$$

$$P(A/E_2) = \frac{2 \times 1}{{}^7C_2} = \frac{2}{21}$$

$$P(A/E_3) = \frac{4 \times 3}{{}^{12}C_2} = \frac{4 \times 3}{66} = \frac{2}{11} \qquad ...(1½)$$

$$\therefore \quad P(E_1/A) = \frac{P(A/E_1).P(E_1)}{P(A/E_1)\ P/(E_1) + P(A/E_2)\ P(E_2) + P(A/E_3).P(E_3)} \qquad ...(1)$$

$$= \frac{\frac{1}{3} \times \frac{1}{5}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{2}{21} \times \frac{1}{3} \times \frac{2}{11}} \qquad ...(½)$$

Marks

$$= \frac{\frac{1}{15}}{\frac{1}{15} + \frac{2}{63} \times \frac{2}{33}}$$

$$= \frac{\frac{1}{15}}{\frac{551}{3465}}$$

$$= \frac{1}{15} \times \frac{3465}{551} = \frac{231}{551}$$ ...(1)

28. Let us suppose that the dealer buys x fans and y sewing machines,

Thus L.P. problem is

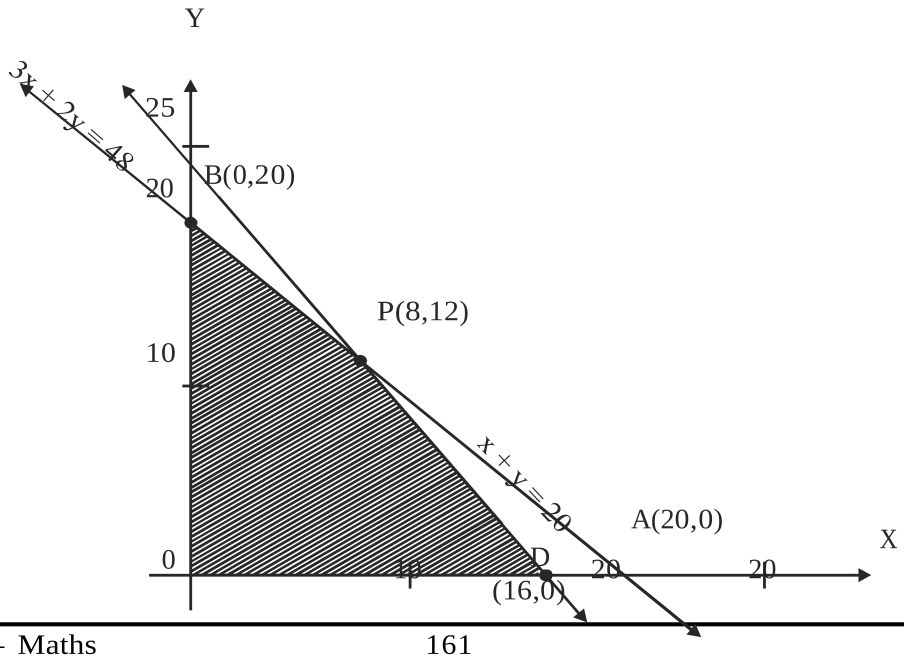
Maximise $Z = 22x + 18y$ ...(1)

subject to constraints,

$$x + y \leq 20$$

$$360x + 240y \leq 5760 \text{ or } 3x + 2y \leq 48$$

$$x \geq 0, \ y \geq 0$$ ...(2)



Marks

For correct graph ...(2)

The feasible region ODPB of the L.P.P. is the shaded region which has the corners O (0, 0), D (16, 0), P (8, 12) and B (0, 20)

The values of the objective function Z at O, D, P and B are :

At O, $Z = 22 \times 0 + 18 \times 0 = 0$

At D, $Z = 22 \times 16 + 18 \times 0 = 352$

At P, $Z = 22 \times 8 + 18 \times 12 = 392 \rightarrow$ Maximum

and At B, $Z = 22 \times 0 + 18 \times 20 = 360$

Thus Z is maximum at $x = 8$ and $y = 12$ and the maximum value of $z =$ Rs 392.

Hence the dealer should purchase 8 fans and 12 sewing machines to obtain maximum profit. ...(1)

29. Let ABC be a right angled triangle with base $BC = x$ and hypotenuse $AB = y$

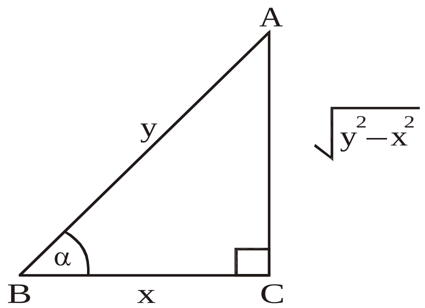
such that

$$x + y = k \text{ where } k \text{ is a constant}$$ ...(½)

Let α be the angle between the base and the hypotenuse.

The area of triangle, $A = \frac{1}{2} BC \times AC$

$$= \frac{1}{2} x \sqrt{y^2 - x^2}$$



...(1½)

Marks

$\therefore \quad A^2 = \frac{x^2}{4}(y^2 - x^2)$

$= \frac{x^2}{4}\left[(k - x)^2 - x^2\right]$

or $\quad A^2 = \frac{x^2}{4}\left[k^2 - 2kx\right] = \frac{k^2x^2 - 2kx^3}{4}$ ..(i)

Differentiating w.r.t. x we get

$2A\frac{dA}{dx} = \frac{2k^2x - 6kx^2}{4}$ ....(ii)

or $\quad \frac{dA}{dx} = \frac{k^2x - 3kx^2}{4A}$ ...(1)

For maximum or minimum,

$\frac{dA}{dx} = 0$

$\Rightarrow \quad \frac{k^2x - 3kx^2}{4} = 0$

$\Rightarrow \quad x = \frac{k}{3}$ ...(1)

Differentiating (ii) w.r.t.x. we get

$2\left(\frac{dA}{dx}\right)^2 + 2A\frac{d^2A}{dx^2} = \frac{2k^2 - 12kx}{4}$

Putting, $\quad \frac{dA}{dx} = 0$ and $x = \frac{k}{3}$, we get

$\frac{d^2A}{dx^2} = \frac{-k^2}{4A} < 0$

Marks

$\therefore$ A is maximum when $x = \frac{k}{3}$ ...(1)

Now $x = \frac{k}{3} \Rightarrow y = k - \frac{k}{3} = \frac{2k}{3}$

$\therefore$ $\cos \alpha = \frac{x}{y} \Rightarrow \cos \alpha = \frac{k/3}{2k/3} = \frac{1}{2}$

$\alpha = \frac{\pi}{3}$ ...(1)

OR

30. Let the length of the tank be x metres and breadth by y metres

$\therefore$ Depth of the tank = 2 metre

$\therefore$ Volume $= x \times y \times 2 = 8$

$xy = 4$

or $y = \frac{4}{x}$ ...(1)

Area of base = xy sq m

Area of 4 walls $= 2\,[2x + 2y] = 4\,(x + y)$

$\therefore$ Cost $C\,(x, y) = 70\,(xy) + 45\,(4x + 4y)$

or $C\,(x, y) = 70 \times 4 + 180\,(x + y)$ ...(1)

$\therefore$ $C(x) = 280 + 180\left(x + \frac{4}{x}\right)$ ...(½)

Now $\frac{dC}{dx} = 180\left(1 - \frac{4}{x^2}\right)$ ...(1)

For maximum or minimum, $\frac{dC}{dx} = 0$

$\therefore$ $180\left(1 - \frac{4}{x^2}\right) = 0$

Marks

or $x^2 = 4$

or $x = 2$ ...(½)

and $$\frac{d^2C}{dx^2} = 180\left(\frac{8}{x^3}\right) > 0$$

$$\left.\frac{d^2C}{dx^2}\right|_{x=2} = 180\left(\frac{8}{8}\right) > 0$$ ...(1)

$\therefore$ C is minimum at $x = 2$

Least Cost = Rs [(280 + 180 (2 + 2)]

= Rs [(280 + 720] = Rs 1000 ...(1)